
Queuing
CEE 320
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Outline

1. Fundamentals
2. Poisson Distribution
3. Notation
4. Applications
5. Analysis
    a. Graphical
    b. Numerical
6. Example

Fundamentals of Queuing Theory

- Microscopic traffic flow
- Arrivals
  - Uniform or random
- Departures
  - Uniform or random
- Service rate
  - Departure channels
- Discipline
  - FIFO and LIFO are most popular
  - FIFO is more prevalent in traffic engineering

Poisson Distribution

- **Count distribution**
  - **Uses discrete values**
  - **Different than a continuous distribution**

$$P(n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

P(n)	=	probability of exactly n vehicles arriving over time t
n	=	number of vehicles arriving over time t
λ	=	average arrival rate
t	=	duration of time over which vehicles are counted

Poisson Ideas

- **Probability of exactly 4 vehicles arriving**
  - P(n=4)
- **Probability of less than 4 vehicles arriving**
  - P(n<4) = P(0) + P(1) + P(2) + P(3)
- **Probability of 4 or more vehicles arriving**
  - P(n≥4) = 1 – P(n<4) = 1 - P(0) + P(1) + P(2) + P(3)

- **Amount of time between arrival of successive vehicles**

$$P(0) = P(h \geq t) = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t} = e^{-qt/3600}$$

Poisson Distribution Example

Vehicle arrivals at the Olympic National Park main gate are assumed Poisson distributed with an average arrival rate of 1 vehicle every 5 minutes. What is the probability of the following:

1. Exactly 2 vehicles arrive in a 15 minute interval?
2. Less than 2 vehicles arrive in a 15 minute interval?
3. More than 2 vehicles arrive in a 15 minute interval?

$$P(n) = \frac{\left(0.20\,\mathrm{veh/min} \times t\right)^n e^{-\left(0.20\,\mathrm{veh/min}\right)t}}{n!}$$

From HCM 2000

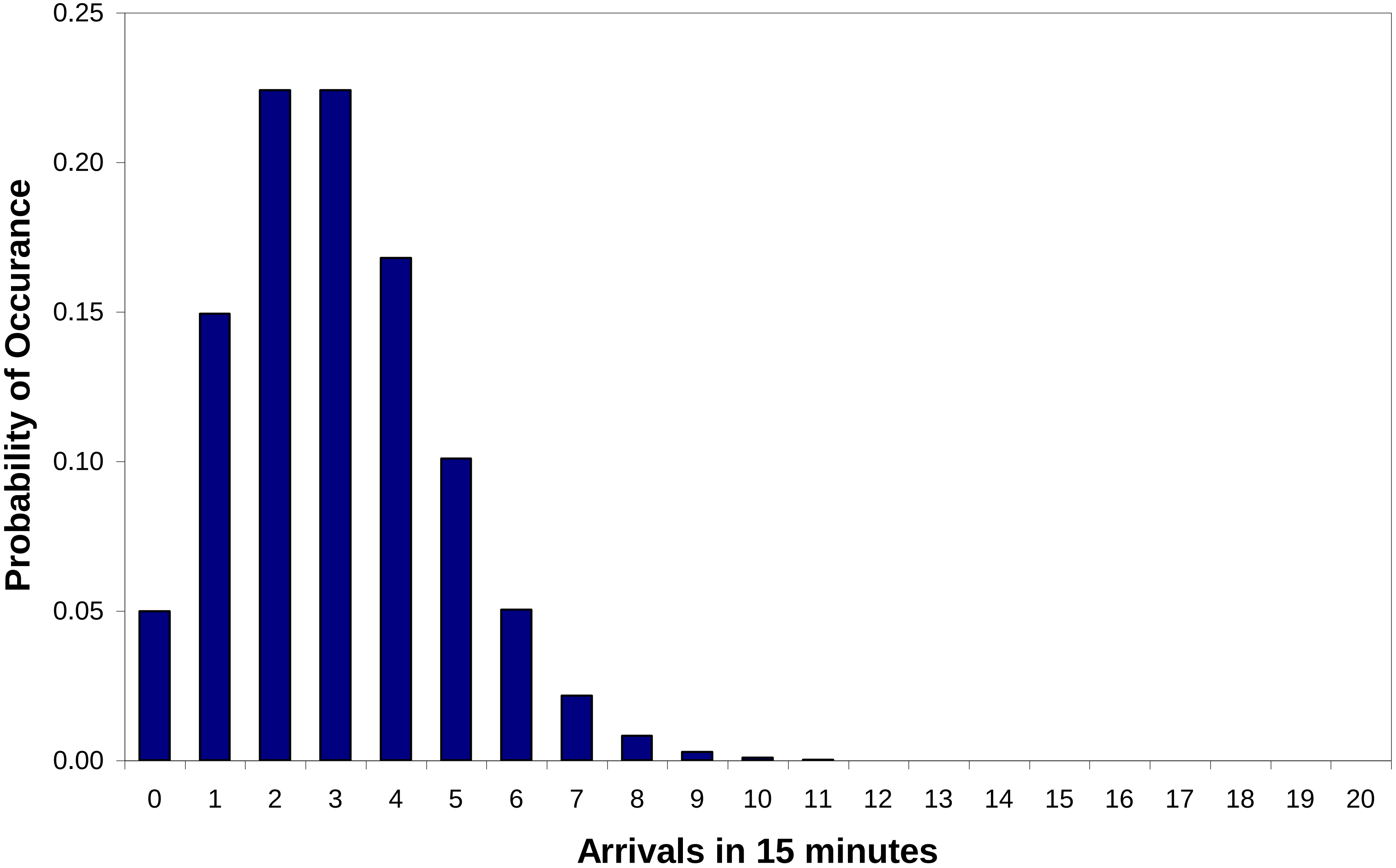
Example Calculations

Exactly 2: $P(2) = \frac{(0.20 \times 15)^2 e^{-(0.20)15}}{2!} = 0.224 = 22.4\%$

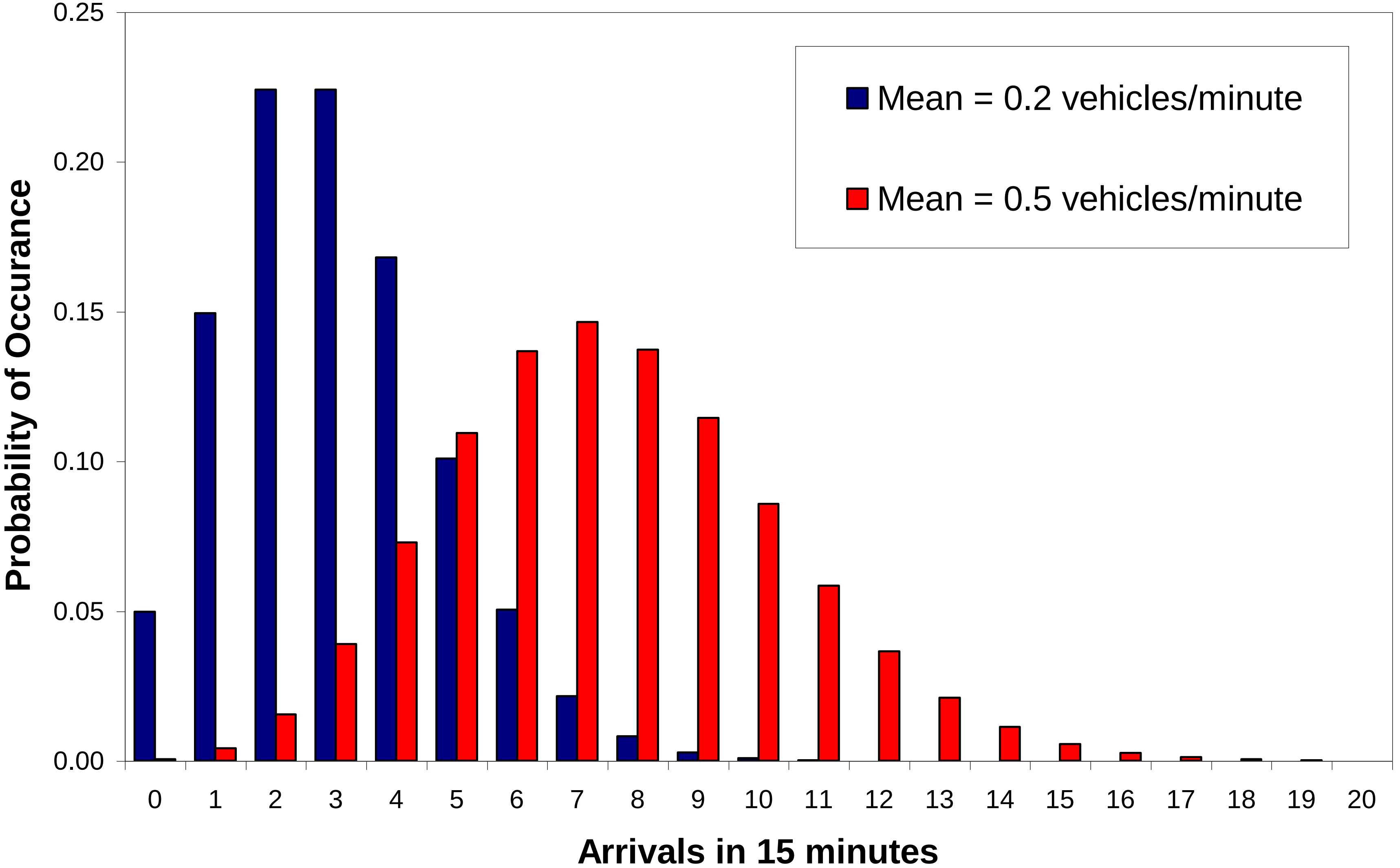
Less than 2: $P(n < 2) = P(0) + P(1)$

More than 2: $P(n > 2) = 1 - (P(0) + P(1) + P(2))$

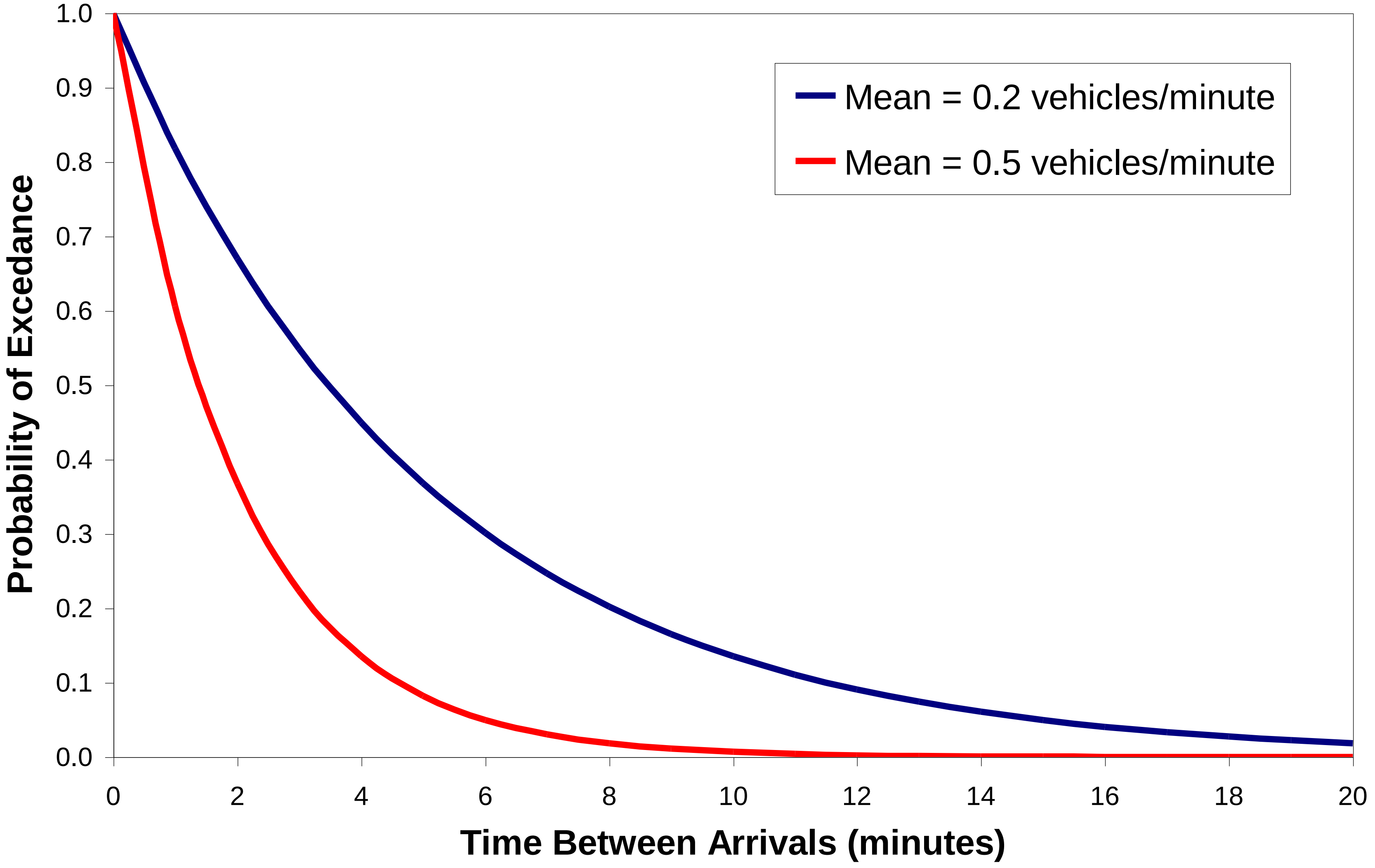
Example Graph



Example Graph



Example: Arrival Intervals



Queue Notation

Arrival rate nature

Number of service channels

$$X \,/\, Y \,/\, N$$

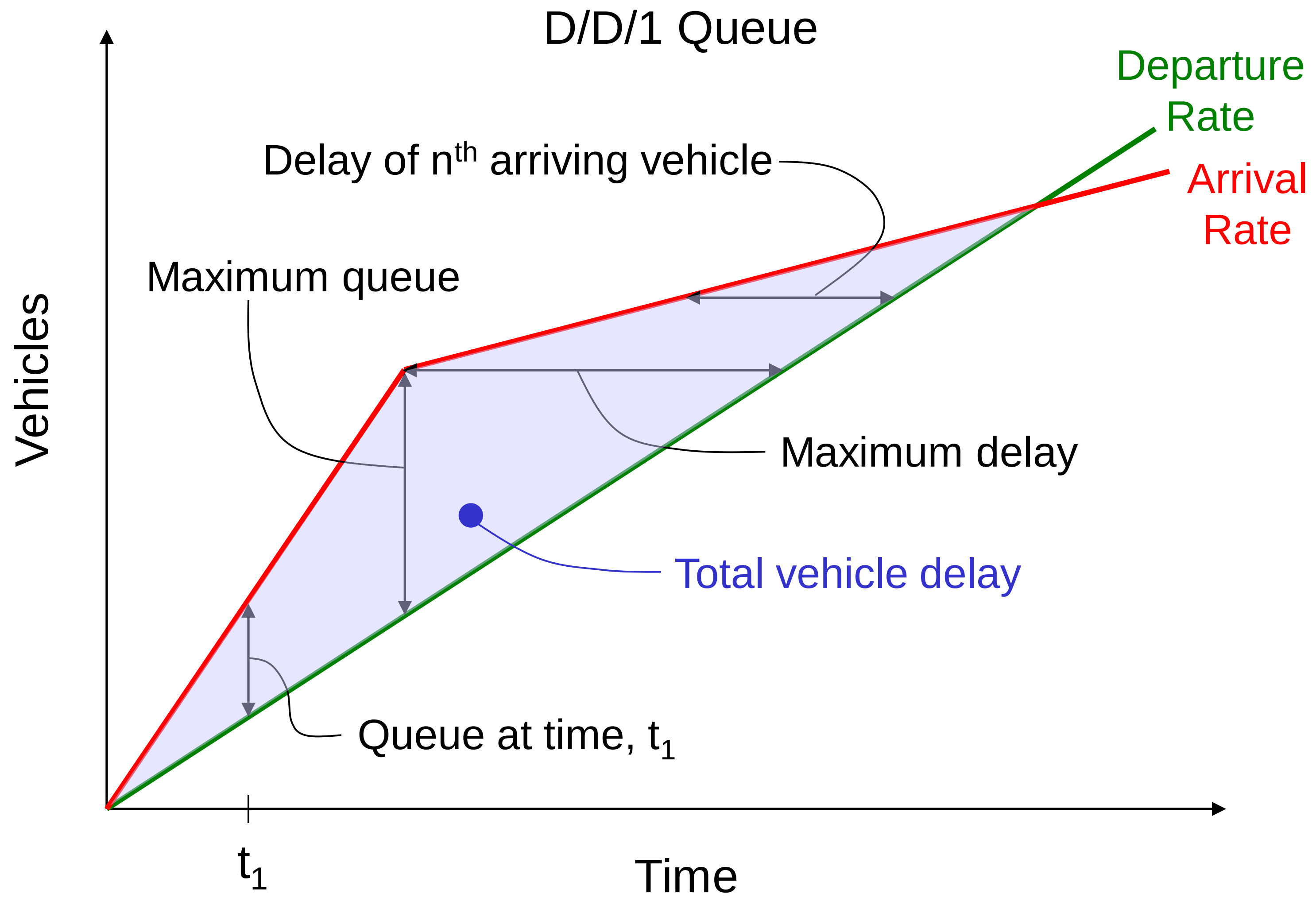
Departure rate nature

- **Popular notations:**
  - **D/D/1, M/D/1, M/M/1, M/M/N**
  - **D = deterministic distribution**
  - **M = exponential distribution**

Queuing Theory Applications

- D/D/1
  - Use only when absolutely sure that both arrivals and departures are deterministic
- M/D/1
  - Controls unaffected by neighboring controls
- M/M/1 or M/M/N
  - General case
- Factors that could affect your analysis:
  - Neighboring system (system of signals)
  - Time-dependent variations in arrivals and departures
    - Peak hour effects in traffic volumes, human service rate changes
  - Breakdown in discipline
    - People jumping queues! More than one vehicle in a lane!
  - Time-dependent service channel variations
    - Grocery store counter lines

Queue Analysis – Graphical



Queue Analysis – Numerical

$$\rho = \frac{\lambda}{\mu} \qquad \rho < 1.0$$

- **M/D/1**
  - **Average length of queue** $$\overline{Q} = \frac{\rho^2}{2(1-\rho)}$$
  - **Average time waiting in queue** $$\overline{w} = \frac{1}{2\mu}\left(\frac{\rho}{1-\rho}\right)$$
  - **Average time spent in system** $$\bar{t} = \frac{1}{2\mu}\left(\frac{2-\rho}{1-\rho}\right)$$

λ = arrival rate μ = departure rate

Queue Analysis – Numerical

$$\rho = \frac{\lambda}{\mu} \qquad \rho < 1.0$$

- **M/M/1**
  - **Average length of queue** $\overline{Q} = \frac{\rho^2}{(1-\rho)}$
  - **Average time waiting in queue** $\overline{w} = \frac{1}{\mu}\left(\frac{\lambda}{\mu-\lambda}\right)$
  - **Average time spent in system** $\bar{t} = \frac{1}{\mu-\lambda}$

λ = arrival rate μ = departure rate

Queue Analysis – Numerical

$$\rho = \frac{\lambda}{\mu} \qquad \rho/N < 1.0$$

- **M/M/N**
  - **Average length of queue** $$\bar{Q} = \frac{P_0 \rho^{N+1}}{N!N}\left[\frac{1}{(1-\rho/N)^2}\right]$$
  - **Average time waiting in queue** $$\bar{w} = \frac{\rho + \bar{Q}}{\lambda} - \frac{1}{\mu}$$
  - **Average time spent in system** $$\bar{t} = \frac{\rho + \bar{Q}}{\lambda}$$

λ = arrival rate μ = departure rate

M/M/N – More Stuff

$\rho = \frac{\lambda}{\mu}$ $\rho/N < 1.0$

– **Probability of having no vehicles**

$$P_0 = \frac{1}{\sum_{n_c=0}^{N-1} \frac{\rho^{n_c}}{n_c!} + \frac{\rho^N}{N!(1-\rho/N)}}$$

– **Probability of having *n* vehicles**

$$P_n = \frac{\rho^n P_0}{n!} \quad \text{for n} \le \text{N}$$

$$P_n = \frac{\rho^n P_0}{N^{n-N} N!} \quad \text{for n} \ge \text{N}$$

– **Probability of being in a queue**

$$P_{n>N} = \frac{P_0 \rho^{N+1}}{N!\,N(1-\rho/N)}$$

λ = arrival rate μ = departure rate

Example 1

You are entering Bank of America Arena at Hec Edmunson Pavilion to watch a basketball game. There is only one ticket line to purchase tickets. Each ticket purchase takes an average of 18 seconds. The average arrival rate is 3 persons/minute.

Find the average length of queue and average waiting time in queue assuming M/M/1 queuing.

Example 2

You are now in line to get into the Arena. There are 3 operating turnstiles with one ticket-taker each. On average it takes 3 seconds for a ticket-taker to process your ticket and allow entry. The average arrival rate is 40 persons/minute.

Find the average length of queue, average waiting time in queue assuming M/M/N queuing.

What is the probability of having exactly 5 people in the system?

Example 3

You are now inside the Arena. They are passing out Harry the Husky doggy bags as a free giveaway. There is only one person passing these out and a line has formed behind her. It takes her exactly 6 seconds to hand out a doggy bag and the arrival rate averages 9 people/minute.

Find the average length of queue, average waiting time in queue, and average time spent in the system assuming M/D/1 queuing.

Primary References

- Mannering, F.L.; Kilareski, W.P. and Washburn, S.S. (2003). *Principles of Highway Engineering and Traffic Analysis*, Third Edition (Draft). Chapter 5

- Transportation Research Board. (2000). *Highway Capacity Manual 2000*. National Research Council, Washington, D.C.